The Noisy Image and the Regulariser and Me

Seb Scott

Supervised by Matthias J. Ehrhardt and Silvia Gazzola

Noisy Images

We want

$x^\star$

We have

y

How to Denoise Images

Given y, find (an approximation of) $x^\star$

$$\hat{x} = \arg\min_{x} L(x, y)$$

Q: What should L be?

$x^\star$

y

$\hat{x}$?

How to Denoise Images

Given y, find (an approximation of) $x^\star$

$$\hat{x} = \arg\min_x \frac{1}{2}\|x-y\|_2^2$$

Q: What should L be?

Candidate:

$$L(x,y) = \frac{1}{2}\|x-y\|_2^2$$

Data-fit: Reconstruction is similar to noisy data

How to Denoise Images

Given y, find (an approximation of) $x^\star$

$$\hat{x} = \arg\min_{x} \frac{1}{2}\|x - y\|_2^2$$

Q: What should L be?

$x^\star$ $\quad$ y $\quad$ $\hat{x}$

How to Denoise Images

Given y, find (an approximation of) $x^\star$

$$\hat{x} = \arg\min_{x} L(x, y)$$

Q: What should L be?

Candidate:

$$L(x, y) = \frac{1}{2}\|x - y\|_2^2 + ?$$

How to Denoise Images

Given y, find (an approximation of) $x^\star$

$$\hat{x} = \arg\min_x L(x, y)$$

Q: What should L be?

Regularisation Parameter: Weighs how important R(x) is

Candidate:

$$L(x, y) = \frac{1}{2}\|x - y\|_2^2 + \alpha R(x)$$

Regulariser: Penalises a noisy image

Outline of Talk

Given y, find (an approximation of) $x^\star$

$$\hat{x} = \arg\min_x \frac{1}{2}\|x - y\|_2^2 + \alpha\, R(x)$$

Q: Good choice of R?

- Examples
- General properties

Q: How to choose α

- Good choice matters
- Finding a good choice

Examples of Regularisers $R(x)$

Given y, find (an approximation of) $x^\star$

$$\hat{x} = \arg\min_{x} \frac{1}{2}\|x-y\|_2^2 + \alpha\, R(x)$$

Q: Good choice of R?

Q: How to choose α

$x^\star$

y

Examples of Regularisers $R(x)$

Given y, find (an approximation of) $x^\star$

$$\hat{x} = \arg\min_x \frac{1}{2}\|x-y\|_2^2 + \alpha R(x)$$

Q: Good choice of R?

Q: How to choose α

$x^\star$

y

2-norm squared

$$R(x) = \|x\|_2^2 = \sum_i |x_i|^2$$

Examples of Regularisers $R(x)$

Given y, find (an approximation of) $x^\star$

$$\hat{x} = \arg\min_x \frac{1}{2}\|x-y\|_2^2 + 0.01\|x\|_2^2$$

Q: Good choice of R?

Q: How to choose α

$x^\star$

y

$\hat{x}$

Examples of Regularisers $R(x)$

Given y, find (an approximation of) $x^\star$

$$\hat{x} = \arg\min_x \frac{1}{2}\|x-y\|_2^2 + \alpha\, R(x)$$

Q: Good choice of R?

Q: How to choose α

$x^\star$

y

1-norm

$$R(x) = \|x\|_1 = \sum_i |x_i|$$

Examples of Regularisers $R(x)$

Given y, find (an approximation of) $x^\star$

$$\hat{x} = \arg\min_{x} \frac{1}{2}\|x - y\|_2^2 + 0.01\|x\|_1$$

Q: Good choice of R?

Q: How to choose α

$x^\star$

y

$\hat{x}$

Examples of Regularisers $R(x)$

Given y, find (an approximation of) $x^\star$

$$\hat{x} = \arg\min_x \frac{1}{2}\|x-y\|_2^2 + \alpha R(x)$$

Q: Good choice of R?

Q: How to choose α

$x^\star$

y

Total Variation (TV)

$$R(x) = TV(x) = \|\nabla x\|_1$$

Examples of Regularisers $R(x)$

Given y, find (an approximation of) $x^\star$

$$\hat{x} = \arg\min_{x} \frac{1}{2}\|x-y\|_2^2 + 0.01\,TV(x)$$

Q: Good choice of R?

Q: How to choose α

$x^\star$

y

$\hat{x}$

Examples of Regularisers $R(x)$

Given y, find (an approximation of) $x^\star$

$$\hat{x} = \arg\min_{x} \frac{1}{2}\|x-y\|_2^2 + \alpha\, R(x)$$

Q: Good choice of R?

Q: How to choose α

$x^\star$

y

Indicator function

$$R(x) = \iota_C(x) = \begin{cases} 0 & x \in C \\ +\infty & x \notin C \end{cases}$$

Examples of Regularisers $R(x)$

Given y, find (an approximation of) $x^\star$

$$\hat{x} = \arg\min_x \frac{1}{2}\|x-y\|_2^2 + 0.01\iota_{[0,255]}(x)$$

Q: Good choice of R?

Q: How to choose α

$x^\star$

y

$\hat{x}$

Outline of Talk (revisited)

Given y, find (an approximation of) $x^\star$

$$\hat{x} = \arg\min_x \frac{1}{2}\|x-y\|_2^2 + \alpha\, R(x)$$

Q: Good choice of R?

Q: How to choose α

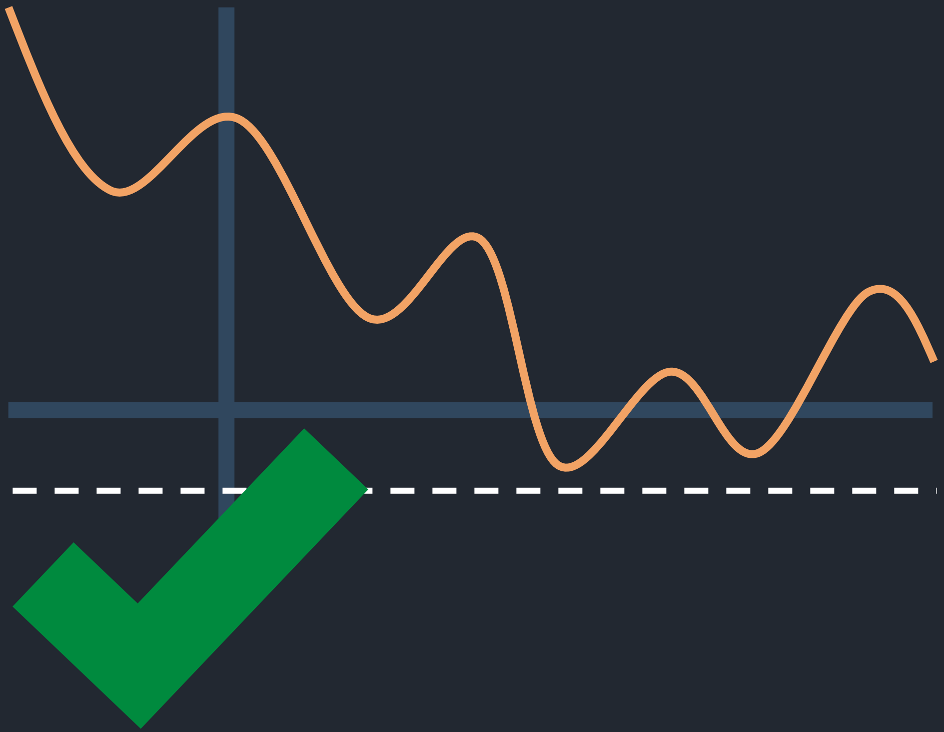
- Examples
- General properties

- Good choice matters
- Finding a good choice

Properties of Regularisers $R(x)$

Consider $R(x)$ that is:

Bounded Below,



Properties of Regularisers $R(x)$

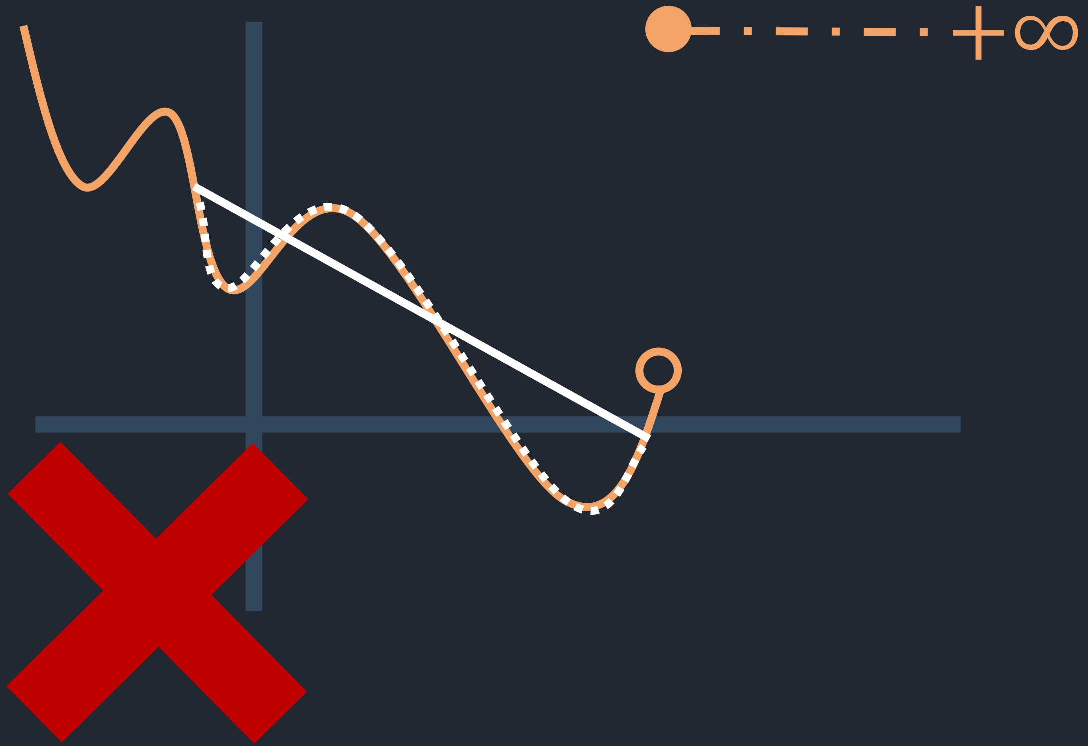
Consider $R(x)$ that is:

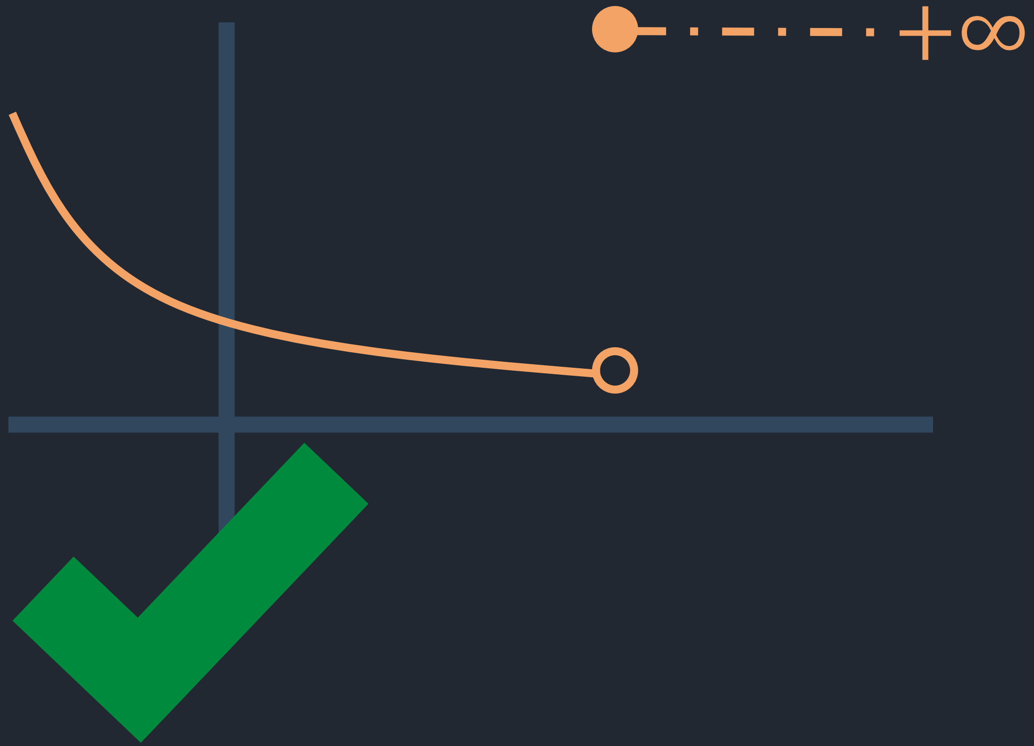
Bounded Below, Proper,

Properties of Regularisers $R(x)$

Consider $R(x)$ that is:

Bounded Below, Proper, Convex,

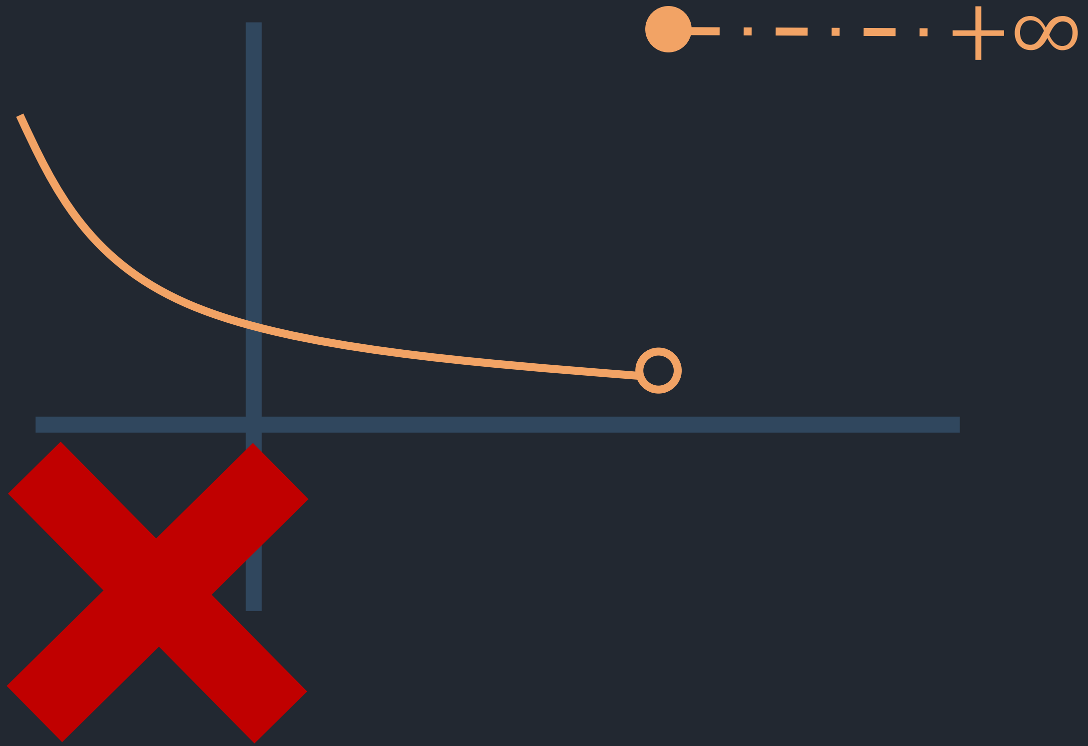


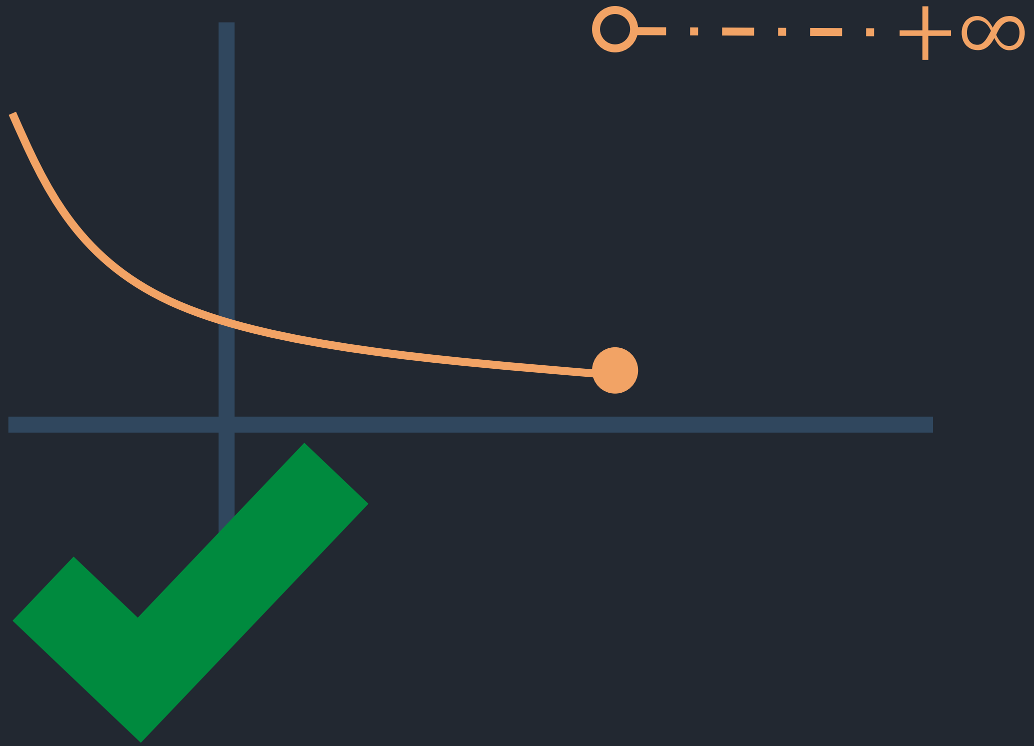


Properties of Regularisers $R(x)$

Consider $R(x)$ that is:

Bounded Below, Proper, Convex, Lower semi-continuous





Properties of Regularisers $R(x)$

Given $\alpha \geq 0$ and $R(x)$ that is

Bounded Below, Proper, Convex, Lower semi-continuous

Then

$$\hat{x} = \arg\min_{x} \frac{1}{2}\|x - y\|_2^2 + \alpha R(x)$$

exists and is unique

Q: How to choose the regularisation parameter α

Choice of α matters

$x^\star$

y

$\hat{x}(\alpha)$

$\alpha = 0.01$

$\alpha = 0.10$

$\alpha = 1.00$

Finding a good α

$x^\star$ 

y 

TASK: Find parameter $\hat{\alpha}$ such that

$$\hat{x}(\hat{\alpha}) = \arg\min_x \frac{1}{2}\|x-y\|_2^2 + \hat{\alpha}\, R(x)$$

is close to $x^\star$

Finding a good α

$x^\star$

y

$$\hat{\alpha} = \arg\min_{\alpha} \frac{1}{2}\|\hat{x}(\alpha) - x^\star\|_2^2$$

$$\hat{x}(\alpha) = \arg\min_{x} \frac{1}{2}\|x - y\|_2^2 + \alpha\, R(x)$$

Finding a good α

Bilevel Optimisation

$$\hat{\alpha} = \arg\min_{\alpha} \frac{1}{2}\|\hat{x}(\alpha) - x^{\star}\|_2^2$$

$$\hat{x}(\alpha) = \arg\min_{x} \frac{1}{2}\|x - y\|_2^2 + \alpha\, R(x)$$

Finding a good α

Bilevel Optimisation

$$\hat{\alpha} = \arg\min_{\alpha} \frac{1}{2}\|\hat{x}(\alpha) - x^{\star}\|_2^2$$

$$\hat{x}(\alpha) = \arg\min_{x} \frac{1}{2}\|x - y\|_2^2 + \alpha\, R(x)$$

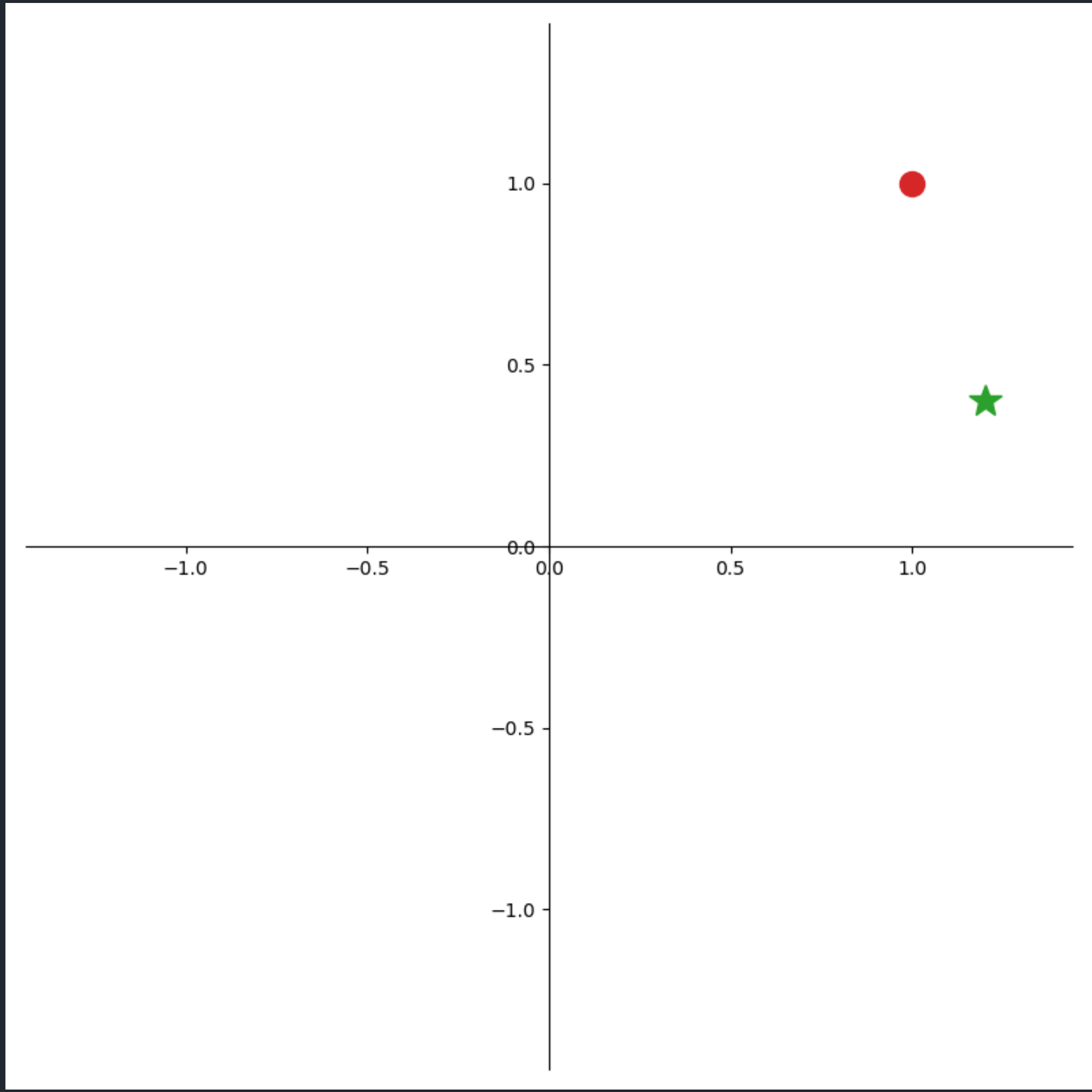
Want $\alpha > 0$

Positivity of $\hat{\alpha}$

Bilevel Optim:

$$\hat{\alpha} = \underset{\alpha}{\arg\min}\ \frac{1}{2}\|\hat{x}(\alpha) - x^\star\|_2^2$$

$$\hat{x}(\alpha) = \underset{x}{\arg\min}\ \frac{1}{2}\|x - y\|_2^2 + \alpha\, R(x)$$

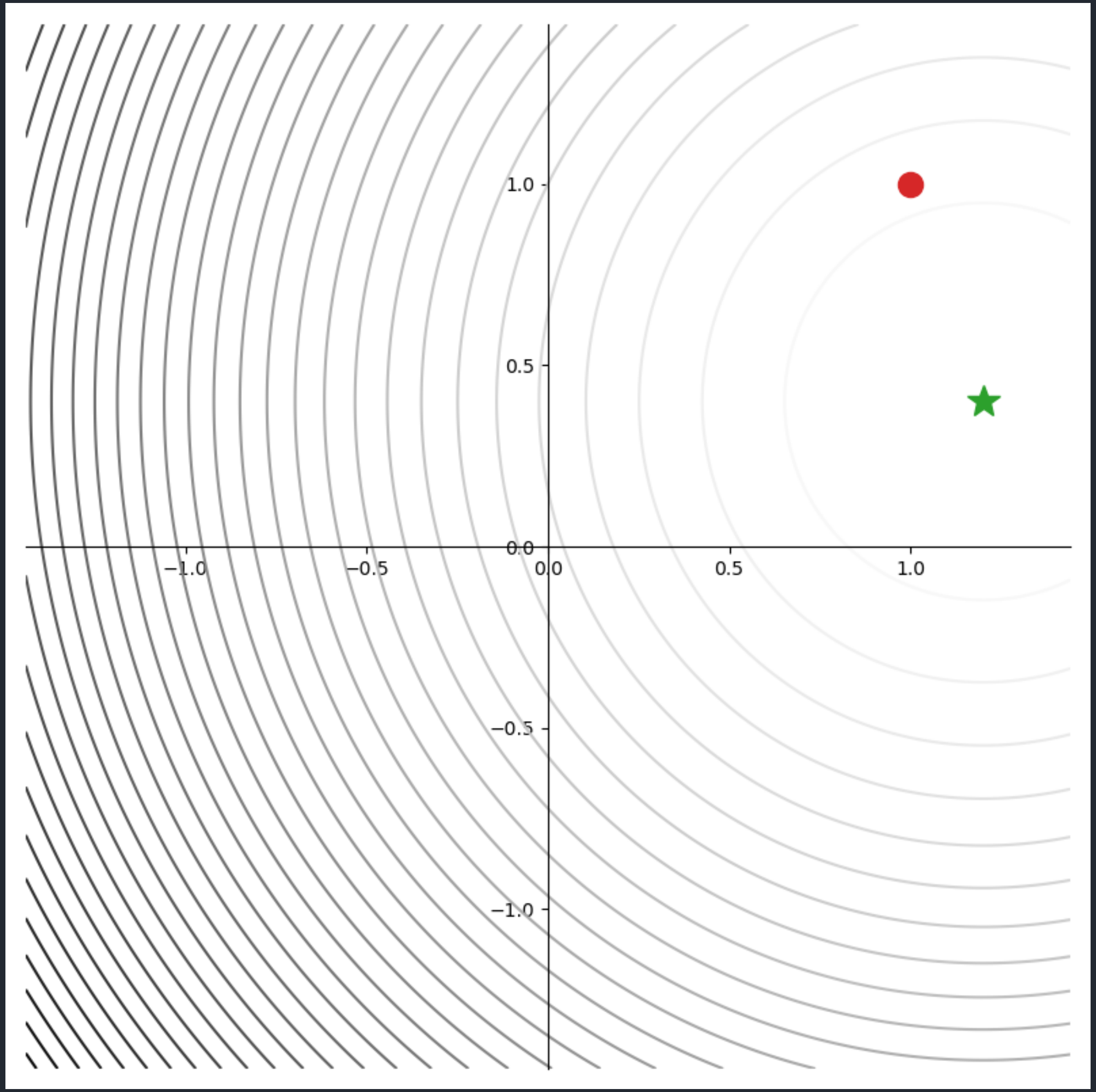


Positivity of $\hat{\alpha}$

Bilevel Optim:

$$\hat{\alpha} = \underset{\alpha}{\arg\min} \ \frac{1}{2}\|\hat{x}(\alpha) - x^\star\|_2^2$$

$$\hat{x}(\alpha) = \underset{x}{\arg\min} \ \frac{1}{2}\|x - y\|_2^2 + \alpha\, R(x)$$



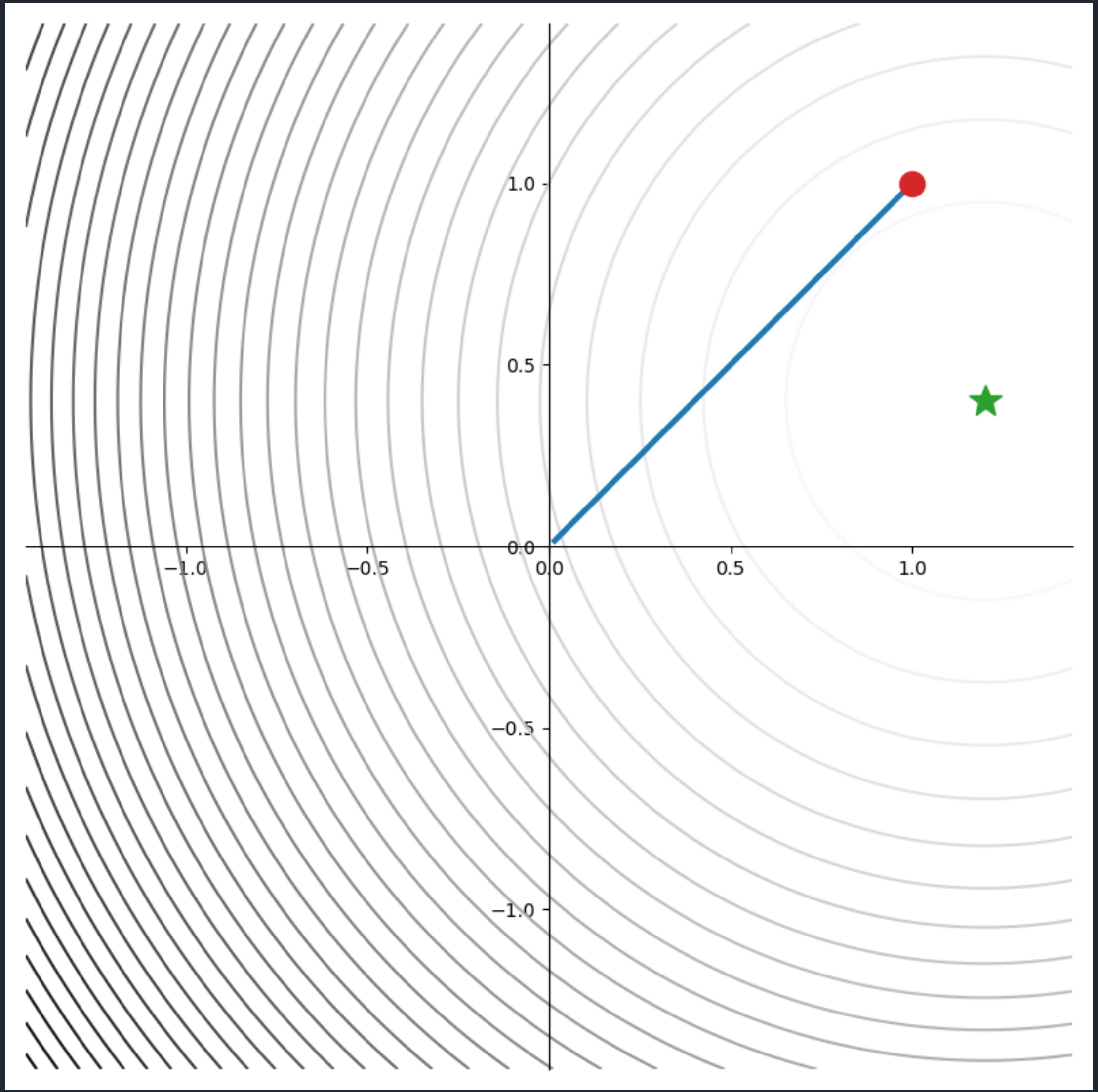
Positivity of $\hat{\alpha}$

Bilevel Optim:

$$\hat{\alpha} = \underset{\alpha}{\arg\min}\ \frac{1}{2}\|\hat{x}(\alpha) - x^\star\|_2^2$$

$$\hat{x}(\alpha) = \underset{x}{\arg\min}\ \frac{1}{2}\|x - y\|_2^2 + \alpha\, R(x)$$

$$R(x) = \|x\|_2^2$$



Positivity of $\hat{\alpha}$

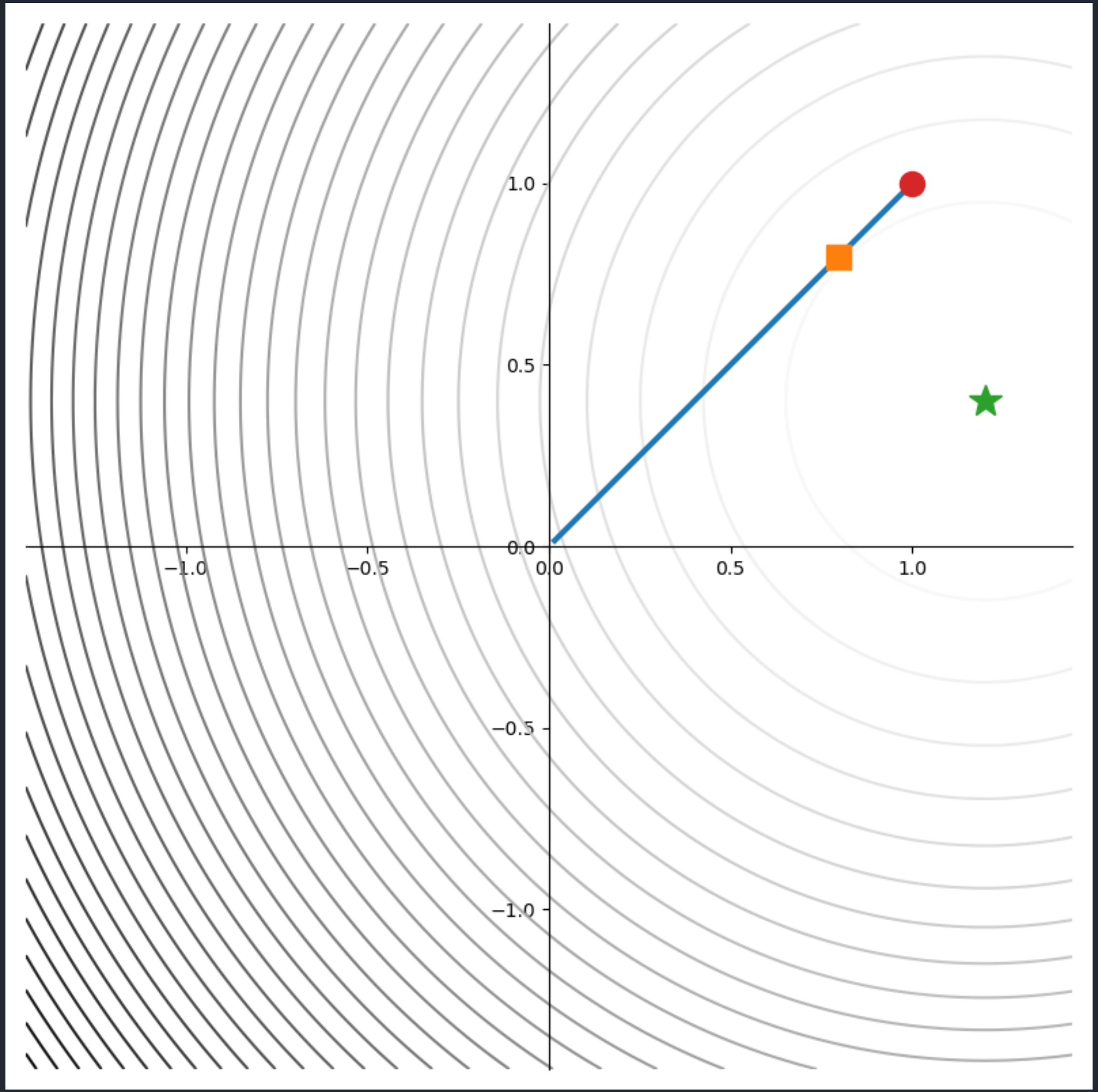
Bilevel Optim:

$$\hat{\alpha} = \arg\min_{\alpha} \frac{1}{2}\|\hat{x}(\alpha) - x^{\star}\|_2^2$$

$$\hat{x}(\alpha) = \arg\min_{x} \frac{1}{2}\|x - y\|_2^2 + \alpha\, R(x)$$

$$R(x) = \|x\|_2^2$$

$$\hat{\alpha} = 0.127$$



Positivity of $\hat{\alpha}$

Bilevel Optim:

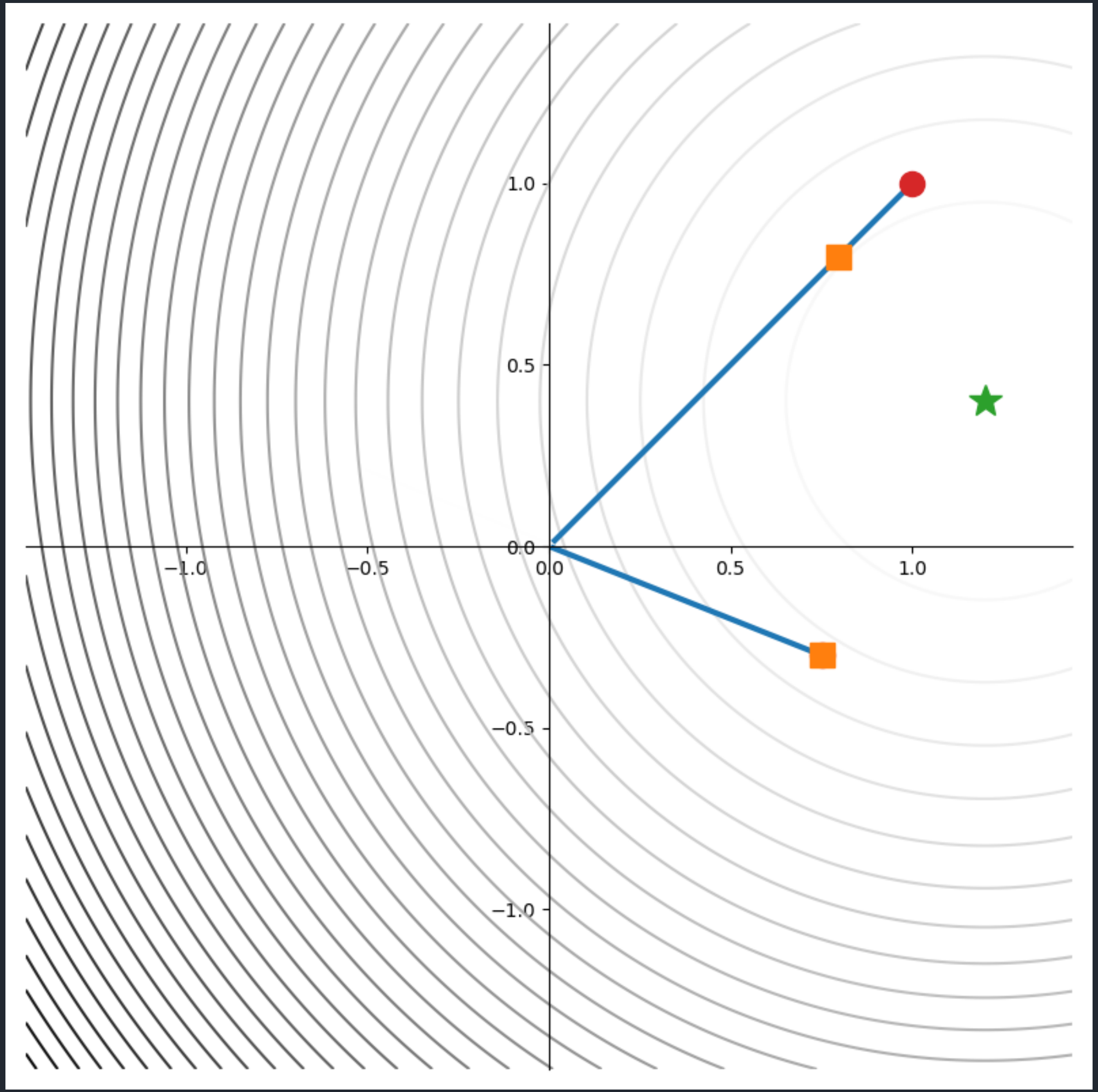
$$\hat{\alpha} = \arg\min_{\alpha} \frac{1}{2}\|\hat{x}(\alpha) - x^{\star}\|_2^2$$

$$\hat{x}(\alpha) = \arg\min_{x} \frac{1}{2}\|x - y\|_2^2 + \alpha\, R(x)$$

$$R(x) = \|x\|_2^2$$

$$\hat{\alpha} = 0.127$$

$$\hat{\alpha} = 0$$



Positivity of $\hat{\alpha}$

Bilevel Optim:

$$\hat{\alpha} = \arg\min_{\alpha} \frac{1}{2}\|\hat{x}(\alpha) - x^{\star}\|_2^2$$

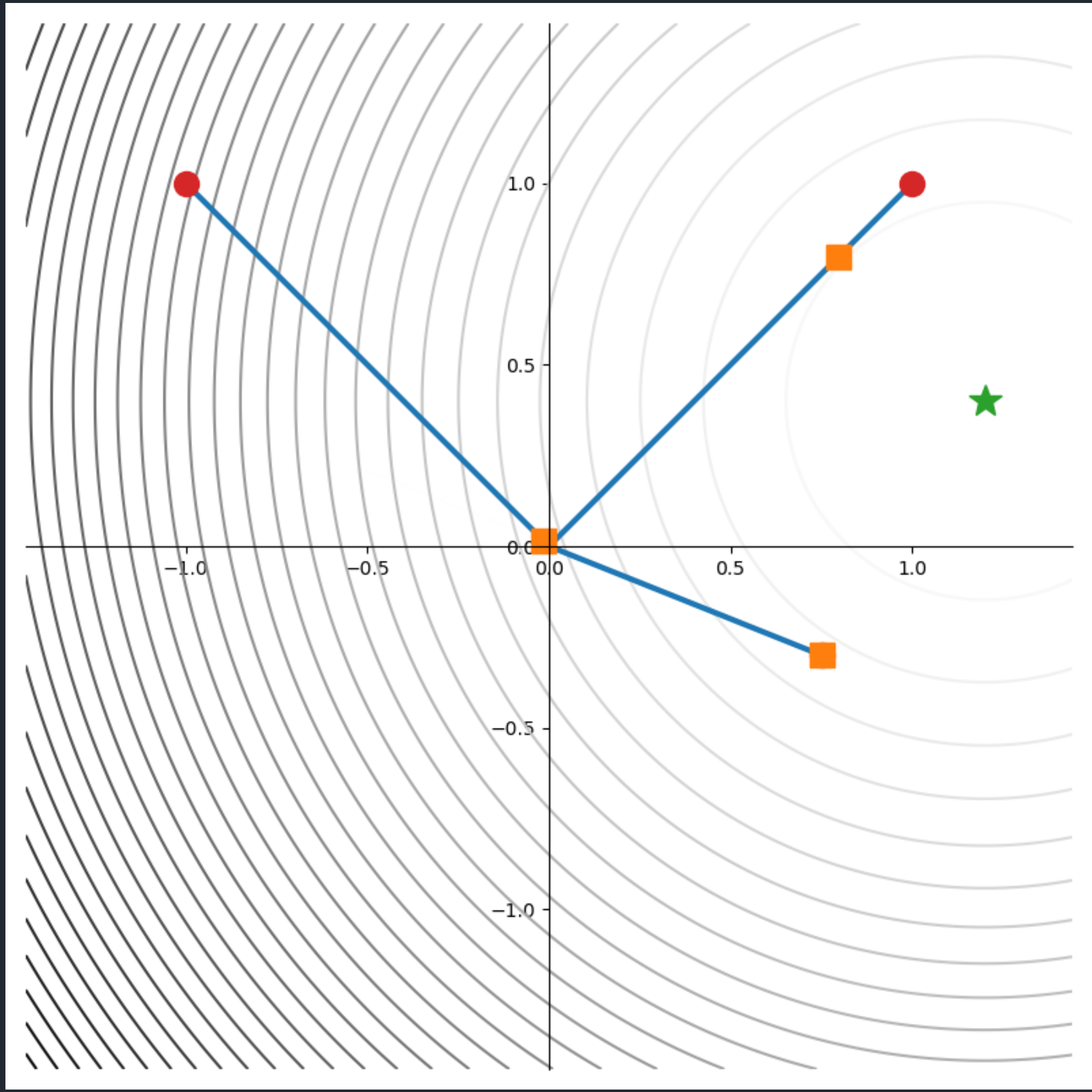
$$\hat{x}(\alpha) = \arg\min_{x} \frac{1}{2}\|x - y\|_2^2 + \alpha\, R(x)$$

$$R(x) = \|x\|_2^2$$

$$\hat{\alpha} = 0.127$$

$$\hat{\alpha} = 0$$

$$\hat{\alpha} = +\infty$$



Positivity of $\hat{\alpha}$

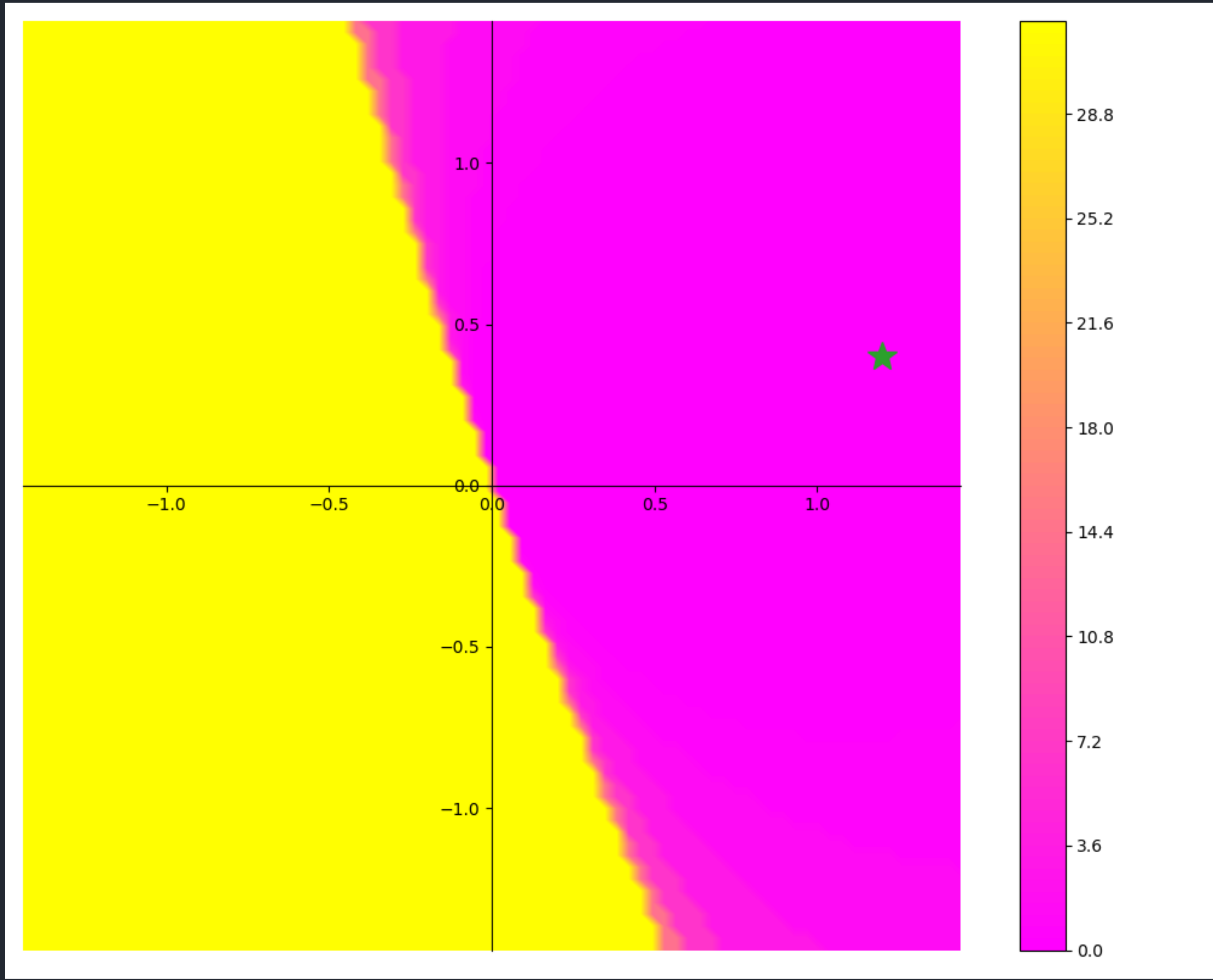
Bilevel Optim:

$$\hat{\alpha} = \underset{\alpha}{\arg\min}\ \frac{1}{2}\|\hat{x}(\alpha) - x^\star\|_2^2$$

$$\hat{x}(\alpha) = \underset{x}{\arg\min}\ \frac{1}{2}\|x - y\|_2^2 + \alpha\, R(x)$$

$$R(x) = \|x\|_2^2$$

Heatmap of $\hat{\alpha}$



Positivity of $\hat{\alpha}$

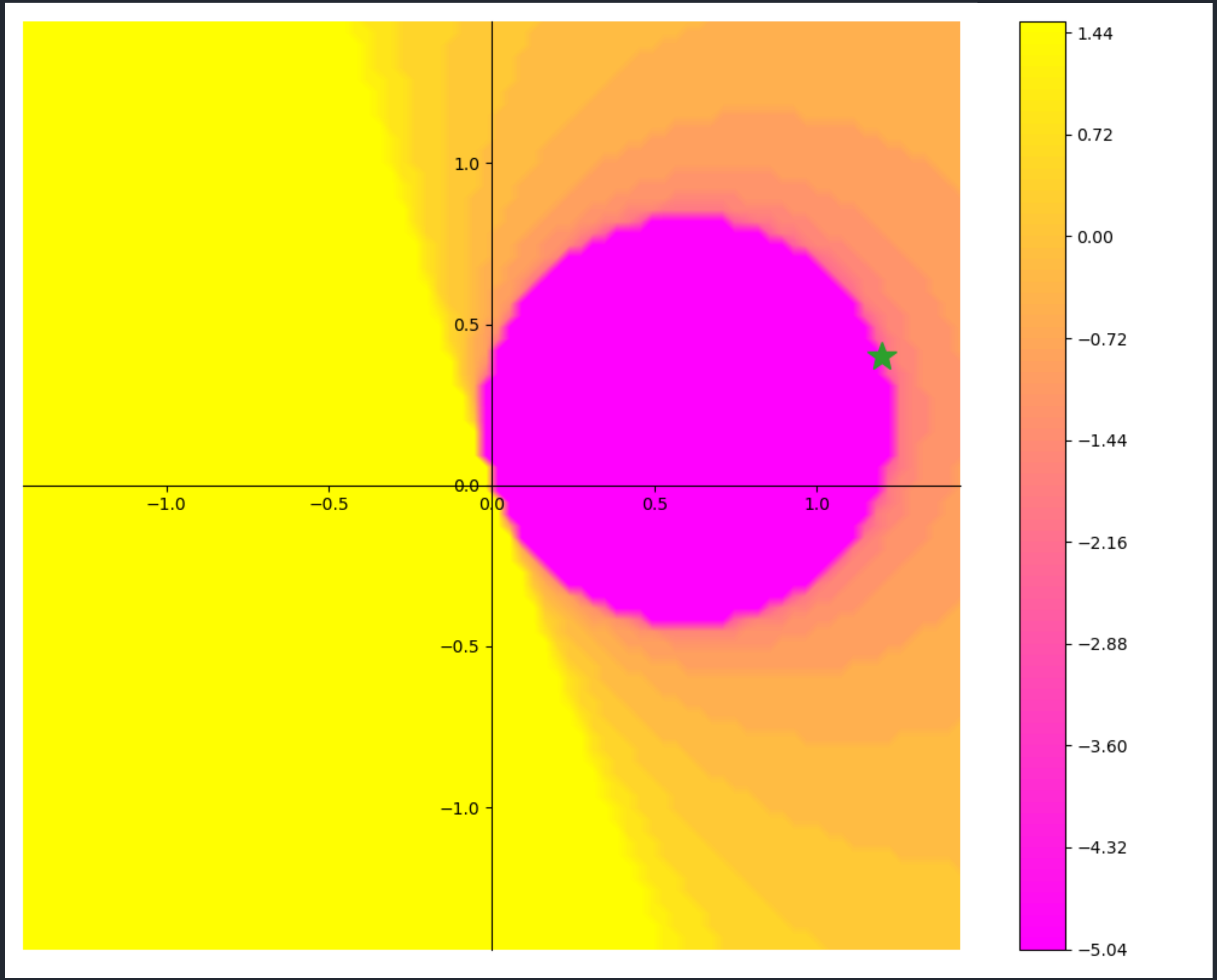
Bilevel Optim:

$$\hat{\alpha} = \arg\min_{\alpha} \frac{1}{2}\|\hat{x}(\alpha) - x^{\star}\|_2^2$$

$$\hat{x}(\alpha) = \arg\min_{x} \frac{1}{2}\|x - y\|_2^2 + \alpha\, R(x)$$

$$R(x) = \|x\|_2^2$$

Heatmap of $\log_{10} \hat{\alpha}$



Positivity of $\hat{\alpha}$

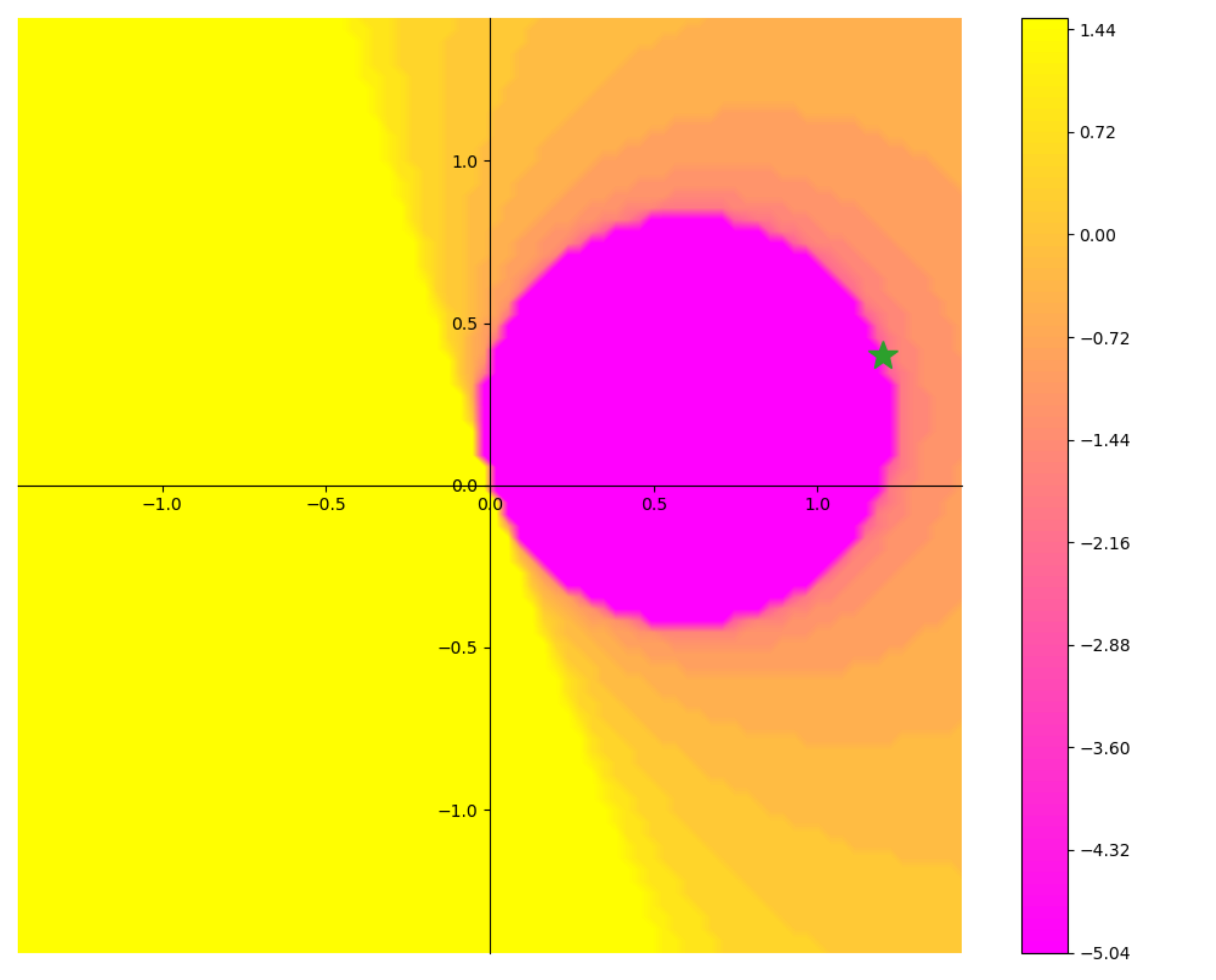
Bilevel Optim:

$$\hat{\alpha} = \arg\min_{\alpha} \frac{1}{2}\|\hat{x}(\alpha) - x^\star\|_2^2$$

$$\hat{x}(\alpha) = \arg\min_{x} \frac{1}{2}\|x - y\|_2^2 + \alpha\, R(x)$$

$$R(x) = \|x\|_2^2$$

Heatmap of $\log_{10} \hat{\alpha}$



Generally believed that

$$R(y) > R(x^\star) \;\Rightarrow\; \hat{\alpha} > 0$$

Positivity of $\hat{\alpha}$

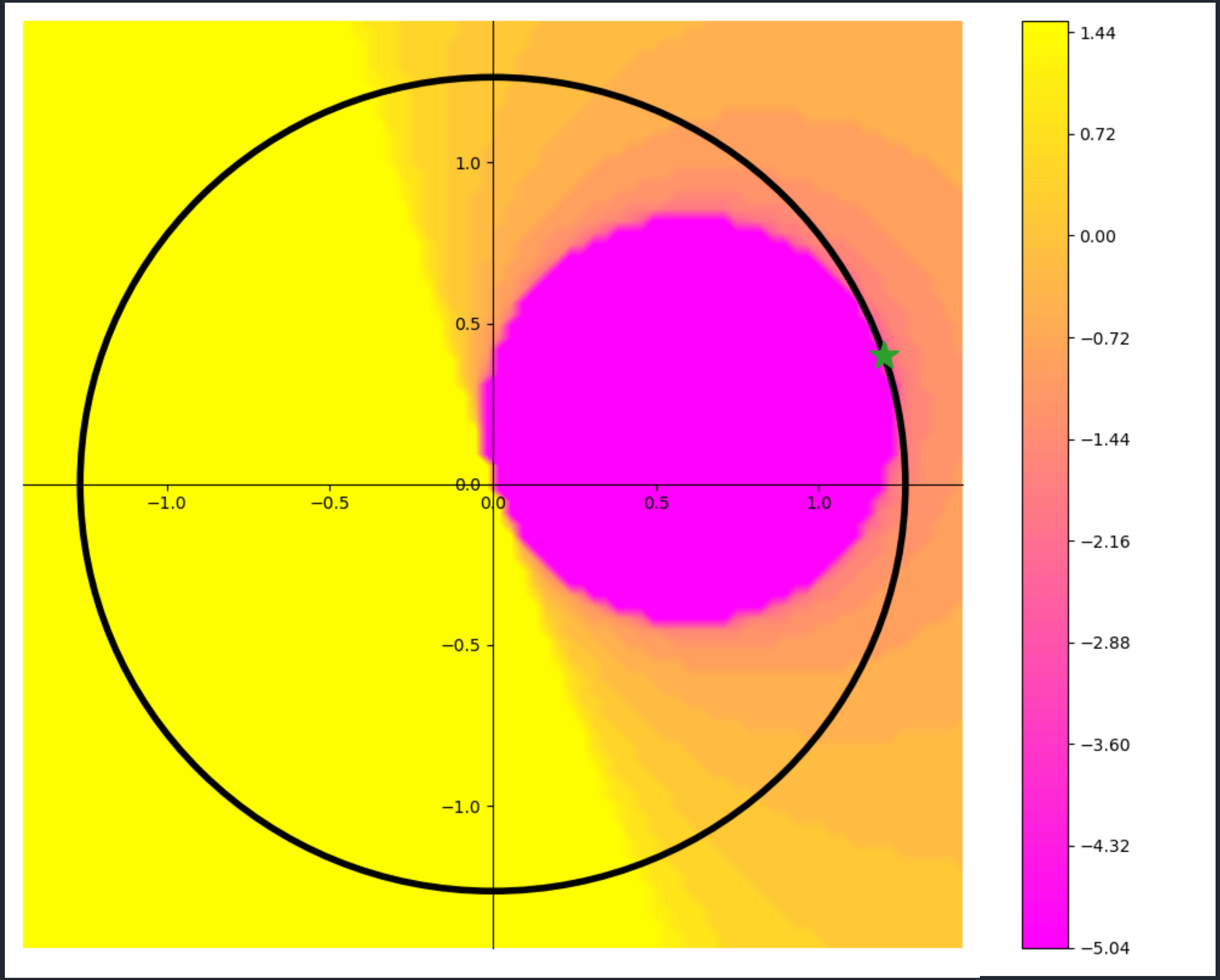
Bilevel Optim:

$$\hat{\alpha} = \arg\min_{\alpha} \frac{1}{2}\|\hat{x}(\alpha) - x^\star\|_2^2$$

$$\hat{x}(\alpha) = \arg\min_{x} \frac{1}{2}\|x - y\|_2^2 + \alpha\, R(x)$$

$$R(x) = \|x\|_2^2$$

Heatmap of $\log_{10} \hat{\alpha}$



Generally believed that

$$R(y) > R(x^\star) \;\Rightarrow\; \hat{\alpha} > 0$$

Positivity of $\hat{\alpha}$

Bilevel Optim:

$$\hat{\alpha} = \arg\min_{\alpha} \tfrac{1}{2}\|\hat{x}(\alpha) - x^\star\|_2^2$$

$$\hat{x}(\alpha) = \arg\min_{x} \tfrac{1}{2}\|x - y\|_2^2 + \alpha\, R(x)$$

Have done:

If $R(x)$ is bounded below, proper, convex, lower semicontinuous and $\acute{\alpha} > 0$ s.t.

$$R(x(\acute{\alpha})) > R(x^\star)$$

then $\hat{\alpha} > 0$

Generally believed that

$$R(y) > R(x^\star) \;\Rightarrow\; \hat{\alpha} > 0$$

Want to do:

If $R(x)$ is bounded below, proper, convex, lower semicontinuous and

$$R(y) > R(x^\star)$$

then $\hat{\alpha} > 0$

Conclusions

Summary

- Denoise images by solving

$$\hat{x} = \arg\min_{x} \tfrac{1}{2}\|x - y\|_2^2 + \alpha\, R(x)$$

- Bilevel optimisation to find optimal $\hat{\alpha}$
- Seems like

$$R(y) > R(x^\star) \quad \Rightarrow \quad \hat{\alpha} > 0$$

Future work

- Prove the thing!
- Finite $\hat{\alpha}$?

Conclusions

Summary

- Denoise images by solving

$$\hat{x} = \arg\min_{x} \frac{1}{2}\|x-y\|_2^2 + \alpha\, R(x)$$

- Bilevel optimisation to find optimal $\hat{\alpha}$
- Seems like

$$R(y) > R(x^\star) \;\Rightarrow\; \hat{\alpha} > 0$$

Future work

- Prove the thing!
- Finite $\hat{\alpha}$?

Thank you!